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QUANTUM COMPUTING FOR MULTI-QUBIT SYSTEMS USING SCHWINGER REPRESENTATION OF ANGULAR MOMENTUM

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Abstract. This paper deals with the features of quantum computing for a system containing a relatively large number of qubits n, which is characterized by a half-integer effective spin S in the Schwinger representation of paired bosons. It was shown that for n = 70, the number of boson states corresponding to lowest 2S + 1 excited energy levels of each of the two harmonic oscillators implementing the two-boson representation of the effective spin S is N = 10^{70} = 1.18 x 10^{21}. Under such conditions, it can be assumed with great precision that for n ≥ 70 all boson states of both harmonic oscillators participate in the implementation of the two-boson representation of the effective spin S. This allows one to apply the quantum field theory methods when performing quantum computations for multi-qubit systems. Another advantage of the Schwinger representation of paired bosons compared to the spinor representation when performing quantum computations for multi-qubit systems is that in the first case the form of the spin projection operators of the effective spin S does not depend on the number of qubits n.

Keywords: *effective spin, harmonic oscillators, bosons, spin projection operators, logical elements.*

Rezumat. Au fost studiate particularitățile calcului cuantic pentru un sistem cu un număr relativ mare de qubiți n, care este caracterizat prin spinul efectiv semiîntreg S în reprezentarea lui Schwinger a bosonilor perechi. S-a demonstrat că pentru n = 70 numărul de stări bosonice care corespund celor mai de jos 2S + 1 niveluri energetice excitate ale fiecăruia dintre cei doi oscilatori armonici, care implementează reprezentarea spinului efectiv S prin intermediul a doi bosoni, ecte N = 10^{70} = 1.18 x 10^{21}. În astfel de condiții, se poate considera cu precizie inaltă că pentru n ⩾ 70 toate stările bosonice ale ambilor oscilatori armonici participă la reprezentarea spinului efectiv S prin intermediul a doi bosoni. Această situație permite aplicarea metodelor teoriei cuantice a câmpului atunci, când se efectuează calcule cuantice pentru sisteme multi-qubit. Un alt avantaj al reprezentării spinului efectiv S prin intermediul bosonilor perechi, în comparație cu reprezentarea spinorică, este că în primul caz forma operatorilor priiecțiilor spinului efectiv S nu depinde de numărul de qubiți n.

Cuvinte cheie: *spin efectiv, oscilatori armonici, bosoni, operatorii proiecțiilor spinului, elemente logice.*

1. Introduction

The first report on the possibility of building a quantum computer as a calculating machine based on quantum-mechanical laws was proposed by Feynman [1]. The fundamental difference between calculations performed quantum and classical computers is that unlike a bit, a quantum bit (cubit) is a linear combination of quantum states |1/2, 1/2> = |0> and |1/2, -1/2> = |1> corresponding to real or effective spin S = 1/2 [2]:

$$|\psi> = \alpha|0> + \beta|1>, \tag{1}$$

where: α and β are complex coefficients satisfying the equality

$$|\alpha|^2 + |\beta|^2 = 1. \tag{2}$$

In the case of real coefficients α and β, the wave function $|\psi>$ is considered as a vector in a linear space in which the orthogonal wave functions |0> and |1> ($<0|0> = 1$, $<1|1> = 1, <0|1> = 0$) play the role of a basis of this space. In the case of complex coefficients α and β, the qubit $|\psi>$ geometrically is presented by the unit vector, which has its origin in the center of the Bloch sphere and the end on its surface. There is an infinite set of points on the surface of Bloch sphere and correspondingly any qubit would be in an infinite set of states. It would seem that from one qubit it would be possible to obtain a set of clones of one arbitrary state $|\psi>$ and carry out measurements of the state of each copy. Then with the probability $|\alpha|^2$ the qubit would be found in the state |0> and with the probability $|\beta|^2$ it would be found in the state |1>. Since $|\alpha|^2$ and $|\beta|^2$ can have infinitely many values, it would be possible to obtain infinitely many information in one qubit. However, in fact it is not so, because during the measurement of the qubit state it can be found only in one from two possible states $|0>$ or $|1>$. Therefore, the scenario described above cannot be implemented due to existence of no-cloning theorem of Nootters and Zurek [3].

Despite the restrictions caused by the no-cloning theorem, the number of publications in the field of quantum computing continues to grow, especially after the discovery of Deutsch-Jozsa [4, 5], Shor [6] and Grover [7,8] algorithms. Practically in all these studies the spinor algebra formalism is used [2] in view of the fact that a system consisting of one or more qubits can be characterized by an effective spin S.

Along with the spinor representation, there are the representations of spin projection operators via creation and annihilation Bose operators, such as Holstein-Primakoff [9] and Dyson-Maleev [10,11] representations. However, these representations are not very convenient in applications. Precisely, in the Holstein-Primakoff representation the spin projection operators S_x and S_y depend on Bose operators in such a way that the last are found under the square root. On the other hand, the use of S_x and S_y operators in the Dyson-Maleev representation leads to the non-Hermitian Hamiltonian. Unlike such behavior of spin projection operators in Holstein-Primakoff and Dyson-Maleev representations, exists the Schwinger representation of angular momentum by means of paired bosons [12]. This representation is free from the deficiencies mentioned and can be successfully applied to quantum computing [13,14]

This paper shows how to simplify quantum computations for n-qubit system (n ⩾ 70) using features of the Schwinger representation of paired bosons for the effective spin S characterizing such a system. Two features of this representation allow us to simplify the implementation of quantum computations. One of them is that the Pauli matrices that define quantum logic elements NOT (or X), Y, Z, and Hadamard gate for a single qubit with effective spin S = 1/2 will also define quantum logic elements for n-qubit systems. This is due to the fact that in the case of these n-qubit systems the spin projection operators of the effective spin S do not depend on the number of qubits n. Another feature of the Schwinger paired bosons representation for the effective spin S is that, starting from some very large number of qubits n, it can be stated with great precision that all boson states of both harmonic oscillators take part in the implementation of this representation. Under these conditions, the quantum field theory methods [15] can be used to perform quantum computations. These methods cannot be used for performing the quantum computing in the case of a small number of qubits n.

2. Features of a n-qubit system due to an effective spin S in the Schwinger paired bosons representation at large values of n

Each multi-qubit system can be characterized by some effective spin S, the value of which depends on the number of cubits n [16]. For n cubits the basis vectors correspond to the eigen states of the spin projection operator S_z related to spin S = 2^{n-1}- 1/2. The value S at different n is presented in Table 1 [16, p. 201].

Table 1

Values of the effective spin S for different numbers n of cubits

n	1	2	3	4	5	6	7	8	9	10	n
S	1/2	3/2	7/2	15/2	31/2	63/2	127/2	255/2	511/2	1023/2	$2^{n-1}-\frac{1}{2}$

According to Table 1, with an increasing number of qubits the effective spin S quickly increase. For example, when n increases from 2 to 3, the effective spin S increases from 3/2 to 7/2, while when n increases from 9 to 10, the effective spin S increases from 511/2 to 1023/2. As a consequence of this, there is also a sharp increase in the dimensionality of the spin projection operator matrices in the spinor representation. For example, the dimensionality of each of the three spin projection operator matrices of the spin S = 1023/2 is 1024 x 1024.

The 2S + 1 spin wave functions related to the effective spin S = 2^{n-1} - 1/2, which characterizes the n-qubit system, is usually given in the spinor representation. However, for quantum computations it is convenient to express these wave functions in the Schwinger paired bosons representation [12]:

$$|S, M_S> = [(S + M_S)!\ (S - M_S)!\ ]^{-1/2}\ (a_1^+)^{S+M_S}(a_2^+)^{S-M_S}\ |0>, \quad (3)$$

where a_i^+(i = 1, 2) and a_i (i = 1, 2) are the operators of creation and annihilation of the bosons related to harmonic oscillators 1 and 2; $|0> = |0>_1 \cdot |0>_2$, $|0>_1$ and $|0>_2$ are the vacuum states related to harmonic oscillators 1 and 2, M_S is the eigen value of the spin projection operator S_z. The spin wave functions $|S, M_S>$ (M_S = S, S-1, S-2, ..., 2-S, 1-S, -S) from (3) can be presented as

$$|S, M_S> = |S + M_S> \cdot |S - M_S>, \quad (4)$$

where:

$$|S + M_S> = [(S + M_S)!\ ]^{-1/2}\ (a_1^+)^{S+M_S}|0>_1,\ |S - M_S> = [(S - M_S)!\ ]^{-1/2}\ (a_2^+)^{S-M_S}|0>_2. \quad (5)$$

The spin projection operators in the Schwinger paired bosons representation using a system of units, in which Planck constant is equal to one, have the form [12]:

$$S_x = (1/2)\ (a_1^+ a_2 + a_2^+ a_1),\ S_y = (i/2)\ (a_2^+ a_1 - a_1^+ a_2),\ S_z = (1/2)\ (a_1^+ a_1 - a_2^+ a_2). \quad (6)$$

3. Results and Discussion

The formulas (6) can be obtained in the different way that it is done in the original Schwinger's work [12]. For this we introduce an unitary spinor operator U_S [14] for which the Hermitian operator U_S^+ is:

$$U_S^+ = \Big(\frac{1}{\sqrt{2S!}} (a_1^+)^{2S} \quad \frac{1}{\sqrt{(2S-1)!}} (a_1^+)^{2S-1} a_2^+ \dots$$
$$\frac{1}{\sqrt{(S+M_S)!(S-M_S)!}} (a_1^+)^{S+M_S} (a_2^+)^{S-M_S} \dots$$
$$\frac{1}{\sqrt{(2S-1)!}} a_1^+ (a_2^+)^{2S-1} \quad \frac{1}{\sqrt{2S!}} (a_2^+)^{2S} \Big)," \tag{7}$$

which satisfies the unitarity condition $U_S^+ U_S$ = **1,** where **1** is the unit operator corresponding to the unit (2S+1) x (2S+1)-matrix. Under the action of the unitary spinor operator U_S, the spin projection operators S_x, S_y, and S_z are transformed as follows:

$$U_S^+ S_x U_S = (1/2)\,(a_1^+ a_2 + a_2^+ a_1)\cdot O_x, \tag{8}$$

$$U_S^+ S_y U_S = (i/2)\,(a_2^+ a_1 - a_1^+ a_2)\cdot O_y, \tag{9}$$

$$U_S^+ S_z U_S = (1/2)\,(a_1^+ a_1 - a_2^+ a_2)\cdot O_z, \tag{10}$$

where: O_x, O_y and O_z are so-called operator loads. In the Ref. [14] it has been proven that $O_x = O_y = O_z$ = **1**.

Taking these equalities into account, formulas (8) – (10) represent a transition from the spinor representation to the Schwinger paired bosons representation using the unitary operator U_S. The right-hand sides of formulas (8) – (10) exactly coincide with the expressions for the spin projection operators S_x, S_y and S_z in the Schwinger paired bosons representation [12]. In the Springer paired bosons representation of an effective spin S the explicit form of the spin projection operators S_x, S_y and S_z should not depend on the value of S. Therefore, taking into account the Table 1, the explicit form of the spin projection operators should also not depend on the number of qubits n. This should lead to a significant simplification of quantum computations for multi-qubit systems. In this case all specific features of multi-qubit systems compared to single-qubit system are determined only by the |S, $M_S >$ spin wave function of a multi-qubit system in the paired bosons representation.

It may be questionable whether in the paired boson representation the quantum logic elements of single- and multi-qubit systems coincide. In the spinor representation such a coincidence is impossible, since in this case any quantum logic element of an n-qubit system is a direct product of n logic elements of a single-qubit systems. However, despite the fact that in the Schwinger paired bosons representation the quantum logic elements do not depend on the number of qubits, the results of the action of these operators on the self functions related to single- and multi-qubit systems are different. This is due to the fact that the eigen vectors of single- and multi-qubit systems are different. Let us demonstrate this with a simple example of a quantum logic element Z = $2S_z$. In the spinor basis, the quantum logic element Z for one qubit has the form

$$Z^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

while, for example, for a three-qubit system the following relation holds:

$$Z^{(3)} = Z_1^{(1)} \otimes Z_2^{(1)} \otimes Z_3^{(1)}. \tag{12}$$

In the Schwinger paired bosons representation, the quantum logic element Z is determined by the operator

$$Z = a_1^+ a_1 - a_2^+ a_2, \tag{13}$$

which retains its form for both a single qubit and in the general case for an n-qubit system. We especially note the absence in this case of the need to calculate the direct product of quantum logic elements, as is required in the case of the spinor representation. In the spinor representation, the eigen functions |S, $M_S >$ and eigen values M_S of the operator S_z are determined by simple equation:

$$S_z \, |S, M_S > = M_S \, |S, M_S >. \tag{14}$$

In this case, the type of the operator S_z, the type of eigen functions |S, $M_S >$ and the eigen values M_S depend on the number of qubits. In contrast to this, in the Schwinger paired bosons representation, the equation (14) has the form:

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ |S, M_S > = M_S \, |S, M_S >, \tag{15}$$

which does not depend on the number of qubits. Although the operator ($a_1^+ a_1$ - $a_2^+ a_2$) does not depend on the number of qubits n, the spin wave functions |S, $M_S >$ and eigen values M_S depend on n. In the particular case S = 3/2, the equations for determining the eigen values M_S have the form:

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ (a_1^+)^3\ |0>_1 = M_S\ (a_1^+)^3\ |0>_1, \tag{16}$$

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ (a_1^+)^2 a_2^+\ |0> = M_S\ (a_1^+)^2 a_2^+\ |0>, \tag{17}$$

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ a_1^+\ (a_1^+)^2 |0> = M_S\ a_1^+ (a_1^+)^2\ |0>, \tag{18}$$

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ (a_2^+)^3\ |0>_2 = M_S\ (a_2^+)^3\ |0>_2. \tag{19}$$

It is easy to show that the solutions of equations (16)-(19) are M_S = 3/2, 1/2, -1/2, -3/2, which, as they should be, coincide with the solutions of equation (14) given in the spinor representation. This particular example demonstrates that although in the Schwinger paired bosons representation the operator S_z does not depend on the number of qubits n, its eigen functions |S, $M_S >$ and eigen values M_S depend on n. Since the quantum logic elements NOT (or X), Y, Z, and Hadamard gate H are defined via Pauli matrices that do not depend on the number of qubits, then in the Schwinger paired bosons representation these logic elements have the form

$$\begin{gathered} X = a_1^+ a_2 + a_2^+ a_1, \\ Y = i\ (a_2^+ a_1 - a_1^+ a_2), \\ Z = (a_1^+ a_1 - a_2^+ a_2), \\ H = \frac{1}{\sqrt{2}} [\ a_1^+ (a_1 + a_2\ ) + a_2^+ (a_1 - a_2)]. \end{gathered} \tag{20}$$

Logic elements (20) have the same form both for one qubit and for multi-qubit systems. The same applies to the quantum logic elements T (or π/8), S (not to be confused with spin) and Φ with the corresponding phase factors in the second term of the first and third of these logic elements:

$$\begin{gathered} T = a_1^+ a_1 + \exp\ (i\ \pi/8) \cdot a_2^+ a_2\,, \\ S = a_1^+ a_1 + i\ a_2^+ a_2\ , \\ \Phi = a_1^+ a_1 + \exp\ (i\ \Phi_0/8) \cdot a_2^+ a_2\,. \end{gathered} \tag{21}$$

The result of the action of operators (20) and (21) on the eigen functions of these operators depends on the number of qubits. This statement is confirmed by the fact that in the case of one qubit, instead of equations (16) - (19) for a two-qubit system, it is necessary to consider the equations

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ a_1^+\ |0>_1 = M_S\ a_1^+\ |0>_1,$$

$$(1/2)\ (a_1^+ a_1 - a_2^+ a_2)\ a_2^+\ |0>_2 = M_S\ a_2^+\ |0>_2, \tag{22}$$

whose solutions are M_S = 1/2, -1/2.

The two-boson representation of the effective spin S involves 2S + 1 states corresponding to the lowest energy levels of each of the harmonic oscillators 1 and 2. For a small number of qubits n, this number of states is negligible compared to the infinite number of states of each of the two harmonic oscillators. For example, for n = 2 (S = 3/2) only the four lowest excited states of each of two harmonic oscillators are involved in the implementation of paired bosons representation of the effective spin:

$$|3/2, 3/2> = |3>_1\ |0>_2,\ \ |3/2, 1/2> = |2>_1\ |1>_2,$$

$$|3/2, -1/2> = |1>_1\ |2>_2,\ \ |3/2, -3/2> = |0>_1\ |3>_2. \tag{23}$$

All other energy levels of harmonic oscillators 1 and 2 must be excluded from consideration.

In the general case, the exclusion of boson states of both harmonic oscillators that do not participate in the implementation of the Schwinger paired bosons representation of the effective spin S is performed using the relationship [17] (p. 267):

$$a_1^+ a_1 + a_2^+ a_2 = 2S\cdot\mathbf{1}, \tag{24}$$

where: **1** is the unit operator represented by the unit (2S+1) x (2S+1)-matrix.

With an increase in the number of cubits n, there is a sharp increase in the number of boson states of each of the quantum harmonic oscillators 1 and 2 participating in the two-boson representation of 2S+1 spin states |S, $M_S >$ (M_S = S, S-1, ..., 1-S, -S). Particularly, for n = 70 we obtain 2S+1 = 1.18 x 10^{21}. Therefore, at n $\geq$ 70 the number of boson states of each of the quantum harmonic oscillators 1 and 2 participating to the paired boson representation of spin states can be considered with high precision equal to infinity. In this case, the methods of quantum field theory [15] can be used to perform quantum computations.

The possibilities of simplifying quantum computing in the case of multi-qubit systems considered in this paper may prove useful in connection with the development of quantum computer technology and the expansion of the range of problems solved with their help. Particularly, the IBM Corporation plans to demonstrate a quantum supercomputer in 2025 by linking together three 3,186-qubit "Kookaburra" processors into a 4,158-qubit system [18]. At n = 4,158, the number of boson states of each of the two harmonic oscillators implementing the Springer paired bosons representation of the effective spin S is infinitely large number $2^{4,158}$, which allows the use of quantum field theory methods to perform quantum computations.

4. **Conclusions**

1. Any n-qubit system can be characterized by an effective spin S = $2^{n-1} - \frac{1}{2}$.

2. In the paired bosons Schwinger representation, the spin projection operators S_x, S_y S_z do not depend on the spin value S and, accordingly, on the number of qubits n.

3. The explicit forms of logical elements NOT (or X), Y, Z, H, T (or π/8), S (not to be confused with spin) and Φ do not depend on the number of qubits n.

4. At a high number of qubits n (n $\geq$ 70), the quantum computations for n-qubit systems can be performed using the quantum field theory methods.

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